©2005. **T. A. Mel'nyk**

ASYMPTOTIC ANALYSIS OF THE SPECTRAL NEUMANN PROBLEM IN THICK MULTI-STRUCTURE OF TYPE 3:1:1

The spectral Neumann problem is considered in a thick multi-structure, which is the union of some three-dimensional domain (the junction's body) and a large number of ε-periodically situated thin cylinders along some curve (the joint zone) on the boundary of junction's body. The asymptotic behaviour (as $\varepsilon \to 0$) of the eigenvalues and eigenfunctions is investigated. Three spectral problems form asymptotics for the eigenvalues and eigenfunctions of this problem, namely, the spectral Neumann problem in junction's body; some spectral problem in a plane domain, which is filled up by the thin cylinders in the limit passage (each eigenvalue of this problem has infinite multiplicity); and the spectral problem for some singular integral operator given on the joint zone. The Hausdorff convergence of the spectrum is proved, the leading terms of asymptotics are constructed (as $\varepsilon \to 0$) and asymptotic estimates are justified for the eigenvalues and the eigenfunctions.

1. Introduction and statement of the problem.

A thick multi-structure (or thick junctions) Ω_ε of type $k : m : d$ is the union of some domain $\Omega_0 \in \mathbb{R}^n$ (junction's body) and a large number of ε-periodically situated thin domains along some manifold (the joint zone) on the boundary of junction's body. Here ε is a small parameter and the type $k : m : d$ of the junction refers to the limiting dimensions of the body, the joint zone, and each attached thin domains. This classification was given by T.A. Mel'nyk and S.A. Nazarov in [4]-[8].

Such multi-structures are prototypes of widely used engineering constructions such as long bridges on supports, frameworks of houses, industrial installations, spaceship grids as well as many other physical and biological systems with very distinct characteristic scales.

Thick junctions have special character of the connectedness, namely, there are points in a thick junction, which are at a short distance of order $\mathcal{O}(\varepsilon)$, but the length of all curves, which connect these points in the junction, is order $\mathcal{O}(1)$. As a result, many new difficulties appear in the asymptotic investigation of boundary value problems in thick multi-structures, for example, the loss of the coercitivity of differential operators in the limit passage as $\varepsilon \to 0$, the absence of extension operators that would be bounded uniformly in ε in the Sobolev space W_2^1, the power behaviour of junction-layer solutions at infinity.

The goal of the study of boundary-value problems in thick multi-structures is to describe the asymptotic behaviour of the solutions as $\varepsilon \to 0$, i.e., when the number of attached thin domains infinitely increases and their thickness vanishes. The reader can find extensive reviews on this theme in [4]-[8].

It follows from these papers that the asymptotic behaviour of the solution to a boundary-value problem in a thick junction essentially depends on the type of this junction. In addition, the type determines the scheme of investigation. Thick multi-structures of types $k : m : d$, where $k-m \neq 1$, are more strongly degenerated junctions and there were no full asymptotic investigations of spectral problems and boundary-value problems in such thick junctions for the present. If we consider some boundary value problem in a thick junction of type $k : m : d$ $(k - m \neq 1)$, then any transmission conditions are not reasonable on the joint zone for the corresponding limiting problems in the Sobolev space W_2^1. This is the principle difficulty in contrast to boundary-value problems in thick junctions of types $k : m : d,\ k - m = 1$.

1.1. Statement of the problem. A model thick junction Ω_ε of type $3 : 1 : 1$ consists of the body

$$\Omega_0 = \{x \in \mathbb{R}^3 : \ |x_1| < h, \ |x_3| < b, \ -\varphi(x') < x_2 < 0\},$$

where $x' = (x_1, x_3)$, and a large number of $\varepsilon-$periodically situated thin cylinders

$$G_j(\varepsilon) = \{x : \ 0 \le x_2 < l, \ \varepsilon^{-1}(x_1 - \varepsilon j, x_3) \in \omega\}, \quad j = 0, \pm 1, \dots, \pm N,$$

which are attached to the body along the segment $I_h = \{x : |x_1| < h, x_2 = x_3 = 0\}$, i.e., $\Omega_\varepsilon = \Omega_0 \cup G(\varepsilon)$, where $G(\varepsilon) = \cup_{j=-N}^{N} G_j(\varepsilon)$. Here $\varphi \in C^\infty([-h,h] \times [-b,b])$, $\varphi \ge 1$; $b > 1$; ω is a plane domain with the smooth boundary and it is symmetric relative to the axis $\{x_1 = 0\}$. In addition, $(0,0) \in \omega \subset \{x' : x_1^2 + x_3^2 < \rho_0 < 1/4\}$. The number N is a large positive integer, therefore the value $\varepsilon = 2h/(2N+1)$ is a small parameter, which characterizes the distance between the neighboring thin cylinders and their thickness.

The following spectral Neumann problem

$$-\Delta_x u(x,\varepsilon) = \lambda(\varepsilon)\, u(x,\varepsilon), \quad x \in \Omega_\varepsilon; \qquad \partial_\nu u(x,\varepsilon) = 0, \quad x \in \partial\Omega_\varepsilon, \tag{1}$$

is considered in this paper. Here $\partial_\nu = \partial/\partial\nu$ is the derivative in the direction of outward normal to the surface $\partial\Omega_\varepsilon$.

A spectral stiff problem in Ω_ε with concentrated masses on the thin cylinders was studied by the author and S.A. Nazarov in [8]. In this problem, passage to the limit as $\varepsilon \to 0$ was accompanied by mass concentration within each thin cylinder and by the infinite increase of the stiffness of the cylinders. On the other hand, presence of a large parameter in the transmission condition at the joint zone leads to some simplification, namely, the corresponding limit spectral problem is reduced to the spectral problem for some singular integral operator J given on the segment I_h. This operator acts in the Hörmander spaces $H_{\log,1}(I_h) \mapsto H_{\log,-1}(I_h)$. Only the low frequency convergence of the eigenvalues and some series of the high frequency convergence of the spectrum were studied in [8].

As distinct from [8], three spectral problems form asymptotics of the eigenvalues and eigenfunctions of problem (1), namely, the spectral Neumann problem in junction's body Ω_0; the spectral problem in the rectangle $D = I_h \times (0,l)$, which is filled up by the thin cylinders in the limit passage (each eigenvalue of this problem has infinite multiplicity); and the spectral problem for the integral operator J.

1.2. The corresponding equivalent operator problem. As usual, $\lambda(\varepsilon)$ is an eigenvalue of the problem (1) if there is a nonzero function (eigenfunction) $u \in \mathcal{H}_\varepsilon := H^1(\Omega_\varepsilon)$ satisfying the integral identity

$$\langle u(\cdot,\varepsilon), v \rangle_\varepsilon = (\lambda(\varepsilon) + 1)\, \big(u(\cdot,\varepsilon), v\big)_{\Omega_\varepsilon}, \tag{2}$$

where $\langle \cdot,\cdot \rangle_\varepsilon$ and $(\cdot,\cdot)_{\Omega_\varepsilon}$ are the standard scalar products in $\mathcal{H}_\varepsilon$ and in $L_2(\Omega_\varepsilon)$ respectively. Define the operator $A_\varepsilon : \mathcal{H}_\varepsilon \longmapsto \mathcal{H}_\varepsilon$ by the formula

$$\langle A_\varepsilon u, v \rangle_\varepsilon = \int_{\Omega_\varepsilon} uv\, dx \quad \text{for all} \quad u, v \in \mathcal{H}_\varepsilon\,. \tag{3}$$

It is easy to verify that this operator is self-adjoint, positive, and compact. Now we can re-write problem (1) as the spectral problem for the operator A_ε :

$$A_\varepsilon u(\cdot,\varepsilon) = (\lambda(\varepsilon) + 1)^{-1} u(\cdot,\varepsilon). \tag{4}$$

Thus, for each fixed ε all eigenvalues of problem (1) can be put in order

$$0 = \lambda_0(\varepsilon) < \lambda_1(\varepsilon) \le \ldots \le \lambda_n(\varepsilon) \le \cdots \to +\infty \quad \text{as} \quad n \to +\infty, \tag{5}$$

with the classical convention of repeated values. The corresponding eigenfunctions can be orthonormalized by the following way

$$\int_{\Omega_0} u_n(x,\varepsilon)\, u_m(x,\varepsilon)\, dx + \int_{G_\varepsilon} u_n(x,\varepsilon)\, u_m(x,\varepsilon)\, dx = \delta_{n,m}, \quad \{n, m\} \in \mathbb{N}_0, \tag{6}$$

where $\delta_{n,m}$ is the Kronecker symbol.

The aim is to study the asymptotic behaviour of the eigenvalues $\{\lambda_n(\varepsilon) :\ n \in \mathbb{N}_0\}$ and eigenfunctions $\{u_n(\cdot,\varepsilon) :\ n \in \mathbb{N}_0\}$ as $\varepsilon \to 0$.

2. Formal asymptotic representations.

By virtue of the minimax principle for eigenvalues it is easy to prove the following estimate

$$\lambda_n(\varepsilon) = \min_{E \in E_n} \max_{0 \ne v \in E} \frac{\int_{\Omega_\varepsilon} |\nabla v|^2\, dx}{\int_{\Omega_\varepsilon} v^2\, dx} \le \frac{(\pi n)^2}{l^2}, \tag{7}$$

where E_n is the set of all subspaces of $\mathcal{H}_\varepsilon$ with the dimension n.

Using inequality (7), condition (6) and the integral identity (2), we deduce the estimates for the eigenfunctions

$$\int_{\Omega_\varepsilon} |\nabla u_n(x,\varepsilon)|^2\, dx = \lambda_n(\varepsilon) \ \le\ \pi^2 n^2 l^{-2}. \tag{8}$$

The asymptotic behaviour of an eigenfunction of problem (1) depend essentially on the energy concentration of the corresponding proper-oscillation. This energy is proportional to the value $\int_{\Omega_\varepsilon} |\nabla u_n(x,\varepsilon)|^2\, dx$. Due to estimate (8) and condition (6), the energy of some proper-oscillation is uniformly bounded with respect to ε and can be concentrated either in the junction's body or in the thin cylinders or uniformly in these sets for ε small enough.

We fix the index n for $\lambda_n(\varepsilon)$ and u_n. In what follows we don't write this index. Having observed that L_2-norm of some bounded function on the thin cylinders is order $\mathcal{O}(\varepsilon^{\frac{1}{2}})$, we propose the following asymptotic representations for the eigenfunction u :

$$u(x,\varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^{\frac{k-1}{2}}\ v^{+}_{\frac{k-1}{2}}(x_1, x_2, \frac{x_1}{\varepsilon} - j, \frac{x_3}{\varepsilon}) \tag{9}$$

in the thin cylinder $G_j(\varepsilon)$ $(j = 0, \pm 1, \ldots, \pm N)$;

$$u(x,\varepsilon) \sim v^{-}_0(x) + \sum_{k=2}^{\infty} \varepsilon^{\frac{k-1}{2}}\ v^{-}_{\frac{k-1}{2}}(x) \tag{10}$$

in the junction's body Ω_0;

$$u(x,\varepsilon) \sim w_0(x_1, \frac{x}{\varepsilon}) + \sum_{k=2}^{\infty} \varepsilon^{\frac{k-1}{2}}\ w_{\frac{k-1}{2}}(x_1, \frac{x}{\varepsilon}) \tag{11}$$

near the joint zone I_h; and for the eigenvalue

$$\lambda(\varepsilon) \sim \sum_{k=1}^{\infty} \varepsilon^{\frac{k-1}{2}} \tau_{\frac{k-1}{2}}. \tag{12}$$

The expansions (9) and (10) are usually called *outer expansions*, the expansion (11) is called *inner expansions*.

REMARK 1. As we will show further, some coefficients of (9)-(12) depend on $\ln\varepsilon$. But, since the power scale $\{\varepsilon^{\frac{k}{2}} : k \in \mathbb{N}\}$ gives the main terms of the asymptotic behaviour, we will not indicate the logarithmic dependence of these coefficients.

2.1. The limiting spectral problem in the junction's body. Let the energy of the corresponding proper-oscillation is concentrated in the junction's body Ω_0, i.e.,

$$\underline{\lim}_{\varepsilon\to 0} \int_{\Omega_0} |\nabla u(x,\varepsilon)|^2 \, dx > 0$$

(we recall the index n is omitted). Since the corresponding values $\lambda(\varepsilon)$ and $\|u(\cdot,\varepsilon)\|_{\mathcal{H}_\varepsilon}$ are bounded with respect to ε, we can choose a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) such that

$$u(\cdot,\varepsilon) \to v^- \neq 0 \ \text{ weakly in } \ H^1(\Omega_0) \ \text{ and } \ \lambda(\varepsilon) \to \mu \ \text{ as } \ \varepsilon \to 0.$$

Passing to the limit in the integral identity (2) with a test function, which vanishes on the thin cylinders and in a neighbourhood of the joint zone, we obtain that μ is an eigenvalue and v^- is the corresponding eigenfunction of the following spectral Neumann problem

$$-\Delta v^-(x) = \mu\, v^-(x), \quad x \in \Omega_0; \qquad \partial_\nu v^-(x) = 0, \quad x \in \partial\Omega_0. \tag{13}$$

Its spectrum σ_{Ω_0} consists of non-decreasing sequence of finite-to-one eigenvalues

$$0 = \mu_0 < \mu_1 \le \ldots \le \mu_n \le \cdots \to +\infty. \tag{14}$$

2.2. Limiting relations in the rectangle $D = (-h,h)\times(0,l)$. Assuming for the moment that the functions $v^+_{\frac{k-1}{2}}$ in (9) are smooth, we write their Taylor series with respect to the first argument at the point $x_1 = j\varepsilon$ and pass the "fast"variables $\xi_1 = \varepsilon^{-1}x_1$, $\xi_3 = \varepsilon^{-1}x_3$. Then (9) takes the form

$$u(\varepsilon,x) \sim \sum_{k=0}^{\infty} \varepsilon^{k-\frac{1}{2}} V^j_{k-\frac{1}{2}}(x_2,\xi_1-j,\xi_3) + \sum_{k=0}^{\infty} \varepsilon^k V^j_k(x_2,\xi_1-j,\xi_3), \tag{15}$$

where

$$V^j_{-\frac{1}{2}} = v^+_{-\frac{1}{2}}(\varepsilon j, x_2, \xi_1-j,\xi_3), \qquad V^j_0 = v^+_0(\varepsilon j, x_2,\xi_1-j,\xi_3), \tag{16}$$

$$V^j_{k-\frac{1}{2}} = v^+_{k-\frac{1}{2}}(\varepsilon j, x_2,\xi_1-j,\xi_3) + \sum_{m=1}^{k} \frac{(\xi_1-j)^m}{m!} \frac{\partial^m v^+_{k-m-\frac{1}{2}}}{\partial x_1^m}(\varepsilon j, x_2,\xi_1-j,\xi_3), \tag{17}$$

$$V^j_k = v^+_k(\varepsilon j, x_2,\xi_1-j,\xi_3) + \sum_{m=1}^{k} \frac{(\xi_1-j)^m}{m!} \frac{\partial^m v^+_{k-m}}{\partial x_1^m}(\varepsilon j, x_2,\xi_1-j,\xi_3). \tag{18}$$

Here and further, the arguments of functions involved in calculations are indicated only if their absence may cause confusion.

We substitute the series (15) and (12) in the equation of problem (1) and in the boundary conditions on the lateral surface $S_j(\varepsilon)$ of the cylinder $G_j(\varepsilon)$. Since the Laplace operator takes the form $\Delta_x = \varepsilon^{-2}\Delta_{\xi'} + \partial^2/\partial x_2^2$ in the variables $\xi' = (\xi_1, \xi_3)$ and x_2, the collection of coefficients of the same powers of ε brings to the following problems

$$\Delta_{\xi'} V^j_{\frac{k-1}{2}} = 0 \quad \text{in } \omega(j) = \{\xi : (\xi_1 - j, \xi_3) \in \omega\}, \quad \partial_{\nu(\xi')} V^j_{\frac{k-1}{2}} = 0 \quad \text{on } \partial\omega(j), \tag{19}$$

for $k = 0, 1, 2, 3$. Here $\partial_{\nu(\xi')}$ is the derivative along the normal $\nu(\xi')$, $\xi' \in \partial\omega(j)$. It follows from (19) that $V^j_{\frac{k-1}{2}}$, $k = 0, 1, 2, 3$, are independent of ξ_1, ξ_3. Next we obtain the following problems

$$-\Delta_{\xi'} V^j_{\frac{3}{2}} = \partial^2_{x_2x_2} V^j_{-\frac{1}{2}} + \tau_0 V^j_{-\frac{1}{2}} \quad \text{in } \omega(j), \quad \partial_{\nu(\xi')} V^j_{\frac{3}{2}} = 0 \quad \text{on } \partial\omega(j); \tag{20}$$

$$-\Delta_{\xi'} V^j_2 = \partial^2_{x_2x_2} V^j_0 + \tau_0 V^j_0 + \tau_{\frac{1}{2}} V^j_{-\frac{1}{2}}(x_2) \quad \text{in } \omega(j), \quad \partial_{\nu(\xi')} V^j_2 = 0 \quad \text{on } \partial\omega(j), \tag{21}$$

where $\partial_{x_i} = \partial/\partial x_i$, $\partial^2_{x_ix_q} = \partial^2/\partial x_i \partial x_q$. The solvability conditions for the problems (20) and (21) respectively read as follows

$$-|\omega|\partial^2_{x_2x_2} V^j_{-\frac{1}{2}} = \tau_0 |\omega| V^j_{-\frac{1}{2}}, \qquad -|\omega|\partial^2_{x_2x_2} V^j_0 = |\omega|\big(\tau_0 V^j_0 + \tau_{\frac{1}{2}} V^j_{-\frac{1}{2}}\big), \quad x_2 \in (0, l) \tag{22}$$

where $|\omega|$ is the Lebesgue measure of the plane domain ω.

Taking into account (16) and the fact that the points εj, $j = 0, \pm 1, \ldots, \pm N$, form an $\varepsilon-$net for the interval $I_h = (-h, h)$, we can extend equations (22), defined initially on $2N+1$ segments, to the entire rectangle $D = (-h, h) \times (0, l)$:

$$-\partial^2_{x_2x_2} v^+_{-\frac{1}{2}}(x^0) = \tau_0 v^+_{-\frac{1}{2}}(x^0), \quad -\partial^2_{x_2x_2} v^+_0(x^0) = \tau_0 v^+_0(x^0) + \tau_{\frac{1}{2}} v^+_{-\frac{1}{2}}(x^0), \quad x^0 \in D. \tag{23}$$

Here $x^0 = (x_1, x_2)$. In accordance with the Neumann conditions on the external bases of the thin cylinders $G_0(\varepsilon)$, $G_{\pm 1}(\varepsilon)$, ..., $G_{\pm N}(\varepsilon)$, we supplement (23) by the conditions

$$\partial_2 v^+_{-\frac{1}{2}}(x_1, l) = 0, \quad \partial_2 v^+_0(x_1, l) = 0, \quad x_1 \in I_h. \tag{24}$$

For other terms of asymptotic expansions (15) and (12) we can similarly obtain the following recurrence relations

$$-\partial^2_{x_2x_2} V^j_{k+\frac{1}{2}}(x_2) - \tau_0 V^j_{k+\frac{1}{2}}(x_2) = \sum_{m=0}^{k} \big(\tau_{k-m+1} V^j_{m-\frac{1}{2}} + \tau_{k-m+\frac{1}{2}} V^j_m\big), \quad x_2 \in (0, l), \tag{25}$$

$$-\partial^2_{x_2x_2} V^j_k(x_2) - \tau_0 V^j_k(x_2) = \tau_{\frac{1}{2}} V^j_{k-\frac{1}{2}} + \sum_{m=0}^{k-1} \big(\tau_{k-m} V^j_m + \tau_{k-m+\frac{1}{2}} V^j_{m-\frac{1}{2}}\big), \quad x_2 \in (0, l), \tag{26}$$

and the following conditions $\partial_2 V^j_{k+\frac{1}{2}}(l) = 0$, $\partial_2 V^j_k(l) = 0$; $k \in \mathbb{N}_0$. As before we extend these equations, defined initially on $2N+1$ segments, to the interior of D.

2.3. The limiting spectral problem in the rectangle D. Comparing the first term of the outer expansion (9) on the thin cylinders with the first term of the outer expansion

(10) in junction's body, we conclude that $v^+_{-\frac{1}{2}}(x_1,0)=0,\ x_1\in I_h$. In addition, $v^+_{-\frac{1}{2}}$ satisfies the first relations (23) and (24). Thus, $v^+_{-\frac{1}{2}}$ must be either trivial or an eigenfunction of the following problem

$$\begin{array}{c} -\partial^2_{x_2x_2} v(x^0) = \tau_0\, v(x^0), \quad x^0 = (x_1,x_2)\in D; \\ v(x_1,0)=0, \quad \partial_{x_2} v(x_1,l)=0, \quad x_1\in I_h. \end{array} \tag{27}$$

It should be noted that eigenvalues of problem (27) form the sequence

$$\sigma_D = \Big\{ \Big(\frac{\pi(2m-1)}{2l}\Big)^2 : \quad m\in\mathbb{N}\Big\} \tag{28}$$

and each eigenvalue has infinite multiplicity. Indeed, for any function $\psi\in C^\infty(I_h)$ the function $v(x^0)=\psi(x_1)\varphi_m(x_2)$ satisfies the problem (27) if the function

$$\varphi_m(x_2)=\sin\big(\pi\, l^{-1}(m-2^{-1})\, x_2\big), \quad x_2\in(0,l). \tag{29}$$

2.4. The limiting spectral problem for problem (1). Problems (13) and (27) forms the limiting spectral problem for problem (1). Let us write the corresponding operator spectral problem for the limit problem. Consider a Hilbert vector space $\mathcal{V}_0 := L_2(\Omega_0)\times L_2(D)$ with the scalar product

$$\big(\overline{u},\overline{v}\big)_{\mathcal{V}_0} = \int_{\Omega_0} u^{(1)}v^{(1)}\,dx + \int_D u^{(2)}v^{(2)}\,dx^0, \quad \forall\, \overline{u}=\big(u^{(1)},u^{(2)}\big),\ \overline{v}=\big(v^{(1)},v^{(2)}\big)\in\mathcal{V}_0.$$

By $\mathcal{H}_0$ denote the following anisotropic Sobolev vector-space $H^1(\Omega_0)\times\{u\in L_2(D):\ \exists\,\partial_{x_2}u\in L_2(D),\ u|_{I_h}=0\}$ with the scalar product

$$\big(\overline{u},\overline{v}\big)_{\mathcal{H}_0} = \big(u^{(1)}v^{(1)}\big)_{H^1(\Omega_0)} + \int_D \big(\partial_{x_2}u^{(2)}\,\partial_{x_2}v^{(2)} + u^{(2)}v^{(2)}\big)\,dx^0, \quad \forall\,\overline{u},\,\overline{v}\in\mathcal{H}_0.$$

Define the linear operator $A_0:\mathcal{H}_0\mapsto\mathcal{H}_0$ by the formula

$$\big(A_0\overline{u},\overline{v}\big)_{\mathcal{H}_0} = \big(\overline{u},\overline{v}\big)_{\mathcal{V}_0}, \quad \forall\,\overline{u},\,\overline{v}\in\mathcal{H}_0.$$

It is easy to verify that A_0 is bounded, self-adjoint and positive. Since the imbedding $\mathcal{H}_0\subset\mathcal{V}_0$ is not compact, the operator A_0 is non-compact.

The spectral problem $A_0\overline{v}=\big(\mu+1\big)^{-1}\overline{v}$ in $\mathcal{H}_0$ is equivalent to the spectral problem

$$\begin{array}{rcl} -\Delta v^{(1)}(x) &=& \mu\, v^{(1)}(x), \quad x\in\Omega_0, \quad \partial_\nu v^{(1)}(x)=0, \quad x\in\partial\Omega_0; \\ -\partial^2_{x_2} v^{(2)}(x^0) &=& \mu\, v^{(2)}(x^0), \quad x^0\in D, \quad v^{(2)}(x_1,0)=\partial_{x_2}v^{(2)}(x_1,l)=0, \quad x_1\in I_h, \end{array} \tag{30}$$

whose spectrum is the union $\sigma_{\Omega_0}\cup\sigma_D$ (the spectra of problems (13) and (27)). In this paper we assume that $\sigma_{\Omega_0}\cap\sigma_D=\emptyset$.

2.5. Junction-layer problems. Take the thin cylinder $G_0(\varepsilon)$ and introduce the 3-dimensional "rapid"variables $\xi=\varepsilon^{-1}x$. After passage to $\varepsilon=0$, in the coordinates ξ the set $G_j(\varepsilon)$ transforms to the semi-infinite cylinder $\omega\times(0,+\infty)$, and the domain Ω_0 to the half-space $\{\xi:\xi_2<0\}$. The periodic location of the thin cylinders suggests that the functions of junction-layer type should be taken 1-periodic relative to ξ_1. Therefore, we consider the union Π of the semi-cylinder $\Pi_+ = \{\xi : \ \xi' = (\xi_1, \xi_3) \in \omega, \ \xi_2 \geq 0\}$ and the half-layer $\Pi_- = \{\xi : \ |\xi_1| < 1/2, \ \xi_2 < 0\}$ as the basic domain, where problems for the junction layer will be posed. Let us investigate some properties of the solutions to the following junction-layer problem

$$\begin{array}{rcll} -\Delta_\xi Z(\xi) & = & F(\xi), & \xi \in \Pi, \\ \partial_{\nu(\xi)} Z(\xi) & = & B(\xi), & \xi \in S = \partial\Pi_+ \setminus \omega, \\ \partial_{\xi_2} Z(\xi', 0) & = & 0, & \xi' \in \mathbb{R}^2 \setminus \overline{\omega}, \ \xi_1 \in (-\frac{1}{2}, \frac{1}{2}), \\ \partial^k_{\xi_1} Z(-\frac{1}{2}, \xi_2, \xi_3) & = & \partial^k_{\xi_1} Z(+\frac{1}{2}, \xi_2, \xi_3), & \xi_2 < 0, \ \ k = 0, 1, \end{array} \tag{31}$$

where $\nu(\xi) = \big(\nu_1(\xi'), \nu_2(\xi')\big)$ is the outward normal derivative on S. To formulate the existence theorem for problem (31) we introduce the "energy"space $\mathcal{H}(\Pi)$, which is the closure of the space $C^\infty_{0,\sharp}(\overline{\Pi})$ by the norm

$$\|u\|_{\mathcal{H}} = \big(\|\nabla_\xi u\|^2_{L_2(\Pi)} + \|d^{-1} u\|^2_{L_2(\Pi)}\big)^{1/2},$$

where d is a smooth weight function, positive in $\overline{\Pi}$ and it is equal to $\rho \ln \rho$ for $\rho = (\xi_2^2 + \xi_3^2)^{1/2} > 2$ and $\xi_2 < 0$, and to ξ_2 for $\xi_2 > 2$; $C^\infty_{0,\sharp}(\overline{\Pi})$ is a space of smooth function, which are finite with respect to ξ_2, ξ_3 and satisfy the last periodic condition in (31).

A function $Z \in \mathcal{H}(\Pi)$ is called a weak solution to problem (31) if for all functions $v \in \mathcal{H}$ the following integral identity holds: $\int_\Pi \nabla_\xi Z \cdot \nabla_\xi v \, d\xi = \int_\Pi F v \, d\xi + \int_S B v \, d\sigma_\xi$.

LEMMA 1. *If* $dF \in L_2(\Pi)$, $dB \in L_2(S)$, *and*

$$\int_\Pi F(\xi) \, d\xi + \int_S B(\xi) \, d\sigma_\xi = 0, \tag{32}$$

then there exists a weak solution $Z \in \mathcal{H}(\Pi)$ *to problem (31). The solution* Z *is defined up to an additive constant.*

The proof of this lemma repeats the proof of Theorem 1 in [10], Lemma 4.1 ([11]), and Lemma 3.1 ([7]), where similar problems were considered.

REMARK 2. Due to the symmetry of Π, there exists a unique weak solution to problem (31), which is even or odd relative to ξ_1 if so are F and B (see Remark 3.2 in [7]).

Similarly as in [10, 5, 11, 8], we establish the asymptotic properties of solutions to problem (31).

LEMMA 2. *If* $F \in C^\infty_0(\overline{\Pi})$, $B \in C^\infty(\overline{S})$, $B(\xi) = 0$ *for* $\xi_2 \geq R_0 > 0$, *and condition (32) holds, then there exists the unique solution* $Z \in \mathcal{H}(\Pi)$ *to problem (31), which admits the differentiated asymptotic representation*

$$Z(\xi) = \begin{cases} \mathcal{O}(\exp(-\delta_1 \xi_2)) & as \quad \xi_2 \to +\infty \ \ (\delta_1 > 0), \\ a_1 + \mathcal{O}(\rho^{-1}) & as \quad \rho \to \infty. \end{cases} \tag{33}$$

If the functions F *and* B *are odd with respect to* ξ_1, *then the solution* $Z \in \mathcal{H}(\Pi)$ *decays exponentially at infinity.*

In the next sections, we will see that the leading terms of (11) have the form

$$\varepsilon^{\alpha-1} v^+_{\alpha-1}(x_1,0) + \varepsilon^{\alpha-\frac12} v^+_{\alpha-\frac12}(x_1,0) + \varepsilon^{\alpha}\Big(V_\alpha(x_1,0) + W_1(\xi)\,\partial_{x_1} v^+_{\alpha-1}(x_1,0) + {} + |\omega|\,W_2(\xi)\,\partial_{x_2} v^+_{\alpha-1}(x_1,0)\Big)\big|_{\xi=\varepsilon^{-1}x} + \ldots, \quad (\alpha = \frac{1}{2}, \text{ or } \alpha = 1). \quad (34)$$

Substituting (34) in problem (1) and collecting the coefficients of the same power of ε, we arrive problems for the functions W_1 and W_2. So, the function W_2 must be a non-trivial solution of the homogeneous problem (31), the function W_1 must be a solution of problem (31) with right-hand sides $F(\xi) \equiv 0,\ B(\xi) = -\nu_1(\xi')$.

COROLLARY 1. *The homogeneous problem (31) has nontrivial solution W_2, which does not belong to $\mathcal{H}(\Pi)$; this solution has the differential asymptotics*

$$W_2(\xi) = \begin{cases} |\omega|^{-1}\xi_2 + \mathcal{O}(\exp(-\delta_2\xi_2)) & \text{as} \quad \xi_2 \to +\infty, \\ -\pi^{-1}\ln\rho + c_\omega + \mathcal{O}(\rho^{-1}) & \text{as} \quad \rho\to\infty \quad (\xi_2 < 0), \end{cases} \quad (35)$$

and is even relative to ξ_1; moreover, $\int_\omega W_2(\xi',0)\,d\xi' = 0$. Here the constant $\delta_2 > 0$.

Problem (31) with right-hand sides $F(\xi) \equiv 0,\ B(\xi) = -\nu_1(\xi')$ has nontrivial solution W_1, which does not belong to $\mathcal{H}(\Pi)$; this solution has the differential asymptotics

$$W_1(\xi) = \begin{cases} -\xi_1 + \mathcal{O}(\exp(-\delta_1\xi_2)) & \text{as} \quad \xi_2 \to +\infty, \\ \mathcal{O}(\exp(-\delta_1\rho)) & \text{as} \quad \rho\to\infty \quad (\xi_2 < 0), \end{cases} \quad (36)$$

and is odd relative to ξ_1 $(\delta_1 > 0)$.

Proof. Such a nontrivial solution to the homogeneous problem can be found in the form

$$W_2(\xi) = -\pi^{-1}\chi_0(\rho)\ln\rho + |\omega|^{-1}\chi_0(\xi_2)\,\xi_2 + Z_2(\xi),$$

where $\chi_0(t),\ t\in\mathbb{R}$, is a smooth cut-off function equal to 0 on $(-\infty,1]$ and to 1 on $[2,+\infty)$; Z_2 is the solution to problem (31) with right-hand sides

$$F(\xi) = -\pi^{-1}[\Delta,\chi_0](\ln\rho) + |\omega|^{-1}[\Delta,\chi_0](\xi_2), \quad B(\xi)\equiv 0,$$

here $[A,B] = AB - BA$ is the commutator of the operator A and B. The existence of Z_2 follows from Lemma 1 (F has compact support and $\int_\Pi F(\xi)\,d\xi = 0$). It remains to observe that F is even relative to ξ_1 and to apply Remark 2 and Lemmas 2. The absence of a constant term in the asymptotics of W_2 as $\xi_2\to+\infty$ leads to the zero mean over the cross-section of the cylinder Π_+.

Analogously we prove the second part. The solution W_1 with asymptotics (36) is sought in the form $W_1(\xi) = -\chi_0(\xi_2)\,\xi_1 + Z_1(\xi)$, where Z_1 is the solution to problem (31) with right-hand sides $F(\xi) = -\chi_0''(\xi_2)\,\xi_1,\quad B(\xi) = -(1-\chi_0(\xi_2))\,\nu_1(\xi')$.

2.6. A singular solution of the Neumann problem in Ω_0. Substituting (10), (12) in (1) and collecting the coefficients of the powers of ε, we arrive the following problem for the leading term v_0^-

$$\begin{aligned} -\Delta_x v_0^-(x) &= \tau_0\, v_0^-(x), & x\in\Omega_0, \\ \partial_\nu v_0^-(x) &= 0, & x\in\partial\Omega_0\setminus I_h. \end{aligned} \quad (37)$$

It is obvious that the eigenvalues (14) of problem (13) and the corresponding eigenfunctions, which is smooth in $\overline{\Omega_0}$, satisfy (37).

The method of matched asymptotic expansions that we use here requires that the solution of problem (37) and (31) admit similar representations (respectively, near the segment $I_h = \{x : |x_1| < h, \ x_2 = x_3 = 0\}$ and near infinity). Since a logarithmic component is present in the solution to the homogeneous problem (31) (see (35)), we are forced to deal with a solution to (37) that has singularities on I_h. The basic principles of constructing such solutions were formulated in [2, 3, 10]; here we use the approach suggested in [1]. From results of this paper it follows the following lemma.

LEMMA 3. *Let τ_0 be an eigenvalue of the Neumann problem* (13) *with the multiplicity q (the case $q = 0$ does not exclude), $\Phi_1, \dots, \Phi_q$ are the corresponding eigenfunctions orthonormalized in $L_2(\Omega_0)$. Let b be a function from $W^1_\infty(I_h)$ such that $\big(b, \Phi_{i\,|I_h}\big)_{L_2(I_h)} = 0$.*

Then there exists a solution to problem (37) *with the following asymptotics*

$$v_0^-(x) := \mathbb{U}(x; b) = -\pi^{-1} b(x_1) \ln r + (Jb)(x_1) + \\ + \sum_{i=1}^{q} \alpha_i \Phi_{i\,|I_h}(x_1) + \mathcal{O}(r(1 + |\ln r|)), \quad r = \sqrt{x_1^2 + x_3^2} \to 0, \quad (38)$$

where J is the integral operator defined by the formula

$$(Jb)(x_1) = \int_{-h}^{h} (b(s) - b(x_1)) G(x_1, 0, 0; s; \lambda_0)\, ds + \\ + \sum_{\pm} \pm \int_0^{\pm h} (b(s) - b(x_1)) G(x_1, 0; \pm 2h - s; \lambda_0)\, ds + \\ + b(x_1)\{(\pi)^{-1} \ln 2 - g^0_-(x_1 - 0, x_1) - g^0_+(x_1 + 0, x_1)\}. \quad (39)$$

In (39), G is the $4h-$periodic in x_1 solution to the problem

$$\begin{array}{rcll} -\Delta_x G(x; s; \tau_0) - \tau_0\, G(x; s; \tau_0) & = & -\sum_{i=1}^q \mathbf{\Phi}_i(s)\, \mathbf{\Phi}_i(x), & x \in \Omega_0^*, \\ \partial_\nu G(x; s; \tau_0) & = & 0, & x \in \Gamma_0^* \setminus \{s\}, \\ (G(\cdot; s; \tau_0), \mathbf{\Phi}_i)_{\Omega_0^*} & = & 0, \quad i = 1, \dots, q, & s \in I_{2h}, \\ G(x; s; \tau_0) & = & \frac{1}{2\pi |x - s|} + \mathcal{O}(|\ln|x - s||), & x \to s, \end{array} \quad (40)$$

where the symbol s denotes both the point $(s, 0, 0) \in I_{2h}$ and its coordinate on the segment I_{2h}; g is a $4h-$periodic primitive for the function $I_{2h} \ni x_1 \to G(x_1, 0, 0; s; \tau_0)$. Due to the last condition in (40), g can be represented in the form

$$g(x_1, s) = \pm(2\pi)^{-1} \ln|x_1 - s| \pm g^0_\pm(x_1, s) \quad \text{for } \pm x_1 > \pm s.$$

The properties of the operator (39) were studied in [1]; we discuss them briefly. Let $H^s_{\ln}(I_h)$ denote the space of restrictions to I_h of the $4h-$periodic functions belonging to the Hörmander space $H^s_{\ln}(I_{2h})$ with the weight function $\mu_s(\xi) = (1 + \ln|\xi| + |\ln|\xi||)^s$. In other words, the norm in $H^s_{\ln}(I_{2h})$ is glued from the norms

$$\|\gamma\|_{H^s_{\ln}(I_{2h})(\mathbb{R})} = \Big(\int_{\mathbb{R}} \mu_s^2(\xi)\, |\mathcal{F}\gamma(\xi)|^2\, d\xi\Big)^{1/2},$$

where $\mathcal{F}\gamma$ denotes the Fourier transform of γ. The embedding $H^s_{\ln}(I_{2h}) \subset L_2(I_{2h})$ is compact for $s > 0$. From [1] it follows that operator $J : H^{\frac{1}{2}}_{\ln}(I_h) \mapsto H^{-\frac{1}{2}}_{\ln}(I_h) := H^{\frac{1}{2}}_{\ln}(I_h)^*$ is continuous, symmetric (as an operator in $L_2(I_h)$ with the domain of definition $H^{\frac{1}{2}}_{\ln}(I_h)$), discrete (for $\lambda_0 > 0$ large enough its resolvent is a compact operator in $L_2(I_h)$), the eigenvalues of J form a sequence

$$\Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_p \ldots \to -\infty, \tag{41}$$

and corresponding eigenfunctions $\{b_p : p \in \mathbb{N}\}$ belong to $C^\infty(I_h)$ and can be orthonormalized in $L_2(I_h)$.

3. Asymptotic approximations and estimates.

3.1. The case $\tau_0 \in \sigma_{\Omega_0} \setminus \sigma_{\mathbf{D}}$. Let τ_0 be an eigenvalue of problem (13) with multiplicity q; $\Phi_1, \ldots, \Phi_q$ are the corresponding eigenfunctions ortonormalized in $L_2(\Omega_0)$. In this case the first term in (15) is a linear combination

$$v_0^-(x) = \sum_{i=1}^{q} a_i^{(0)} \Phi_i(x), \quad x \in \Omega_0, \tag{42}$$

where the constants $a_i^{(0)}$, $i = 1, \ldots, q$, will be defined further.

Applying the method of matched asymptotic expansions to the leading terms of (10) and (11) and to the leading terms of (11) and (9), we obtain that $v^+_{-\frac{1}{2}}(x_1, 0) = 0$ and $w_0(x_1, \xi) = v_0^-(x_1, 0, 0) = v_0^+(x_1, 0)$. Since τ_0 is not an eigenvalue of problem (27), then $v^+_{-\frac{1}{2}}$ must be trivial. As a result, v_0^+ is the unique solution to the following problem

$$\begin{aligned} -\partial_2^2 v_0^+(x_1, x_2) &= \tau_0 v_0^+(x_1, x_2), \quad (x_1, x_2) \in G_0, \\ v_0^+(x_1, 0) = v_0^-(x_1, 0, 0), &\quad \partial_{x_2} v_0^+(x_1, l) = 0, \quad x_1 \in I_h. \end{aligned} \tag{43}$$

Obviously,

$$v_0^+(x_1, x_2) = \frac{v_0^-(x_1, 0, 0)}{\cos(\sqrt{\tau_0}\, l)} \cos\sqrt{\tau_0}(l - x_2), \quad (x_1, x_2) \in G_0. \tag{44}$$

Matching the next following terms of (9) - (11), we get that all terms in (9), (10), (11) and (12) at $k = 2p$, $p \in \mathbb{N}$, are trivial and these asymptotic expansions are expansions with respect to the nonnegative integral powers of ε.

For the function v_1^-, we obtain the following relations

$$-\Delta_x v_1^- - \tau_0 v_1^- = \tau_1 v_0^-, \quad x \in \Omega_0; \qquad \partial_\nu v_1^- = 0, \quad x \in \partial\Omega_0 \setminus I_h, \tag{45}$$

and because of (34) ($\alpha = 1$), (35), (36), and (44), v_1^- must have the logarithmic singularity

$$\begin{aligned} v_1^-(x) = -\pi^{-1}|\omega| \sqrt{\tau_0} \tan(\sqrt{\tau_0}\, l)\, \gamma_0(x_1) \ln r \, + \\ + |\omega| \sqrt{\tau_0} \tan(\sqrt{\tau_0}\, l)\, \gamma_0(x_1) \left(\pi^{-1} \ln\varepsilon + c_\omega\right) + V_1(x_1, 0) \quad \text{as} \quad r \to 0, \end{aligned} \tag{46}$$

where $\gamma_0(x_1) = v_0^-(x_1, 0, 0) = v_0^+(x_1, 0) = \sum_{i=1}^q a_i^{(0)} \Phi_i|_{I_h}$, $x_1 \in I_h$.

Comparing problem (45) with problem (37) and using the representation (38) with $\alpha_i = 0$, $i = 1, \ldots, q$, we can state that there exists such a singular solution

$$v_1^-(x) = |\omega| \sqrt{\tau_0} \tan(\sqrt{\tau_0}\, l)\, \mathbb{U}(x; \gamma_0), \quad x \in \Omega_0, \tag{47}$$

to problem (46) if

$$\tau_1\,\overline{a}^{(0)} = -\,2\,|\omega|\,\sqrt{\tau_0}\,\tan(\sqrt{\tau_0}\,l)\,\mathbb{M}\,\overline{a}^{(0)}, \qquad (48)$$

where $\overline{a}^{(0)} = \big(a_1^{(0)}, \dots, a_q^{(0)}\big)$ and $\mathbb{M} = \big\{(\Phi_i\,,\Phi_j)_{I_h},\ \ i,j = 1,\dots,q\big\}$ is the Gram matrix, which is symmetric and nonnegative.

From the spectral problem (48) we define τ_1 in (12) and the constants $\{a_i^{(0)}\}$ in (42). The spectral problem (48) has q eigenvalues and we assume that they are simple, i.e., $\tau_1^{(1)} < \dots < \tau_1^{(q)}$; the corresponding eigenvectors $\overline{a}_i^{(0)} = \big(a_{i1}^{(0)}, \dots, a_{iq}^{(0)}\big)$ $i = 1,\dots,q$, can be orthonormalized by the following way $\overline{a}_i^{(0)} \cdot \overline{a}_j^{(0)} = \delta_{ij},\ i,j = 1,\dots,q$. So, let τ_1 be an eigenvalue of (48), $\overline{a}^{(0)} = \big(a_1^{(0)}, \dots, a_q^{(0)}\big)$ is the corresponding normalized eigenvector. Then the function v_0^- is defined and $\|v_0^-\|_{L_2(\Omega_0)} = 1$.

The singular solution v_1^- has the asymptotics

$$v_1^-(x) \sim |\omega|\,\sqrt{\tau_0}\,\tan(\sqrt{\tau_0}\,l)\,\Big(\big(J\gamma_0\big)(x_1) - \pi^{-1}\gamma_0(x_1)\,\ln r + \mathcal{O}(r(1+\ln r))\Big), \quad r \to 0$$

(see Lemma 3). Comparing this asymptotics with (46) and taking into account the matching principle, we deduce the second boundary condition

$$V_1(x_1,0) = |\omega|\,\sqrt{\tau_0}\,\tan(\sqrt{\tau_0}\,l)\,\Big(\big(J\gamma_0\big)(x_1) - \gamma_0(x_1)\big(\pi^{-1}\ln\varepsilon + c_\omega\big)\Big)$$

for the equation (26) at $k=1$ and unique determine the function V_1.

As result, we have defined the leading terms of the outer expansions (9), (10), the inner expansion (34) ($\alpha = 1$), and the expansion (12) and can construct a global asymptotic approximation $U(\cdot,\varepsilon)$ belonging to $H^1(\Omega_\varepsilon)$:

$$\begin{aligned} U(x,\varepsilon) &= (1-\chi(r/\varepsilon))\,\big(v_0^-(x) + \varepsilon v_1^-(x)\big) + \\ &+ \chi(r)\Big(v_0^+(x_1,0) + \varepsilon\,\big(V_1(x_1,0) + W_1(x/\varepsilon)\,\partial_1 v_0^+(x_1,0) + |\omega|\,W_2(x/\varepsilon)\,\partial_2 v_0^+(x_1,0)\big)\Big) - \\ &- \chi(r)\big(1-\chi(\tfrac{r}{\varepsilon})\big)\Big(\gamma_0(x_1) + \varepsilon|\omega|\sqrt{\tau_0}\tan(\sqrt{\tau_0}\,l)\big((J\gamma_0)(x_1) - \frac{1}{\pi}\gamma_0\,\ln r\big)\Big),\ x\in\Omega_0, \end{aligned} \qquad (49)$$

$$\begin{aligned} U(x,\varepsilon) &= v_0^+(x_1,x_2) + \varepsilon\Big(V_1(x_1,x_2) + Y_1(x_1/\varepsilon)\,\partial_1 v_0^+(x_1,x_2) + \\ &+ \chi(x_2)\sum_{i=1}^{2}\big(|\omega|^{\delta_{i,2}}W_i(\tfrac{x}{\varepsilon}) - Y_1(\tfrac{x_1}{\varepsilon})\delta_{i,1} - \varepsilon^{-1}x_2\delta_{i,2}\big)\partial_i v_0^+(x_1,0)\Big), \quad x\in G_\varepsilon, \end{aligned} \qquad (50)$$

where χ is a smooth cut-off function such that $\chi(t) = 1$ if $|t| < R_0 := 4^{-1}\min\{l, b, \min\varphi\}$, and $\chi(t) = 0$ if $|t| > 2R_0$; $Y_1(t) = -t + [t+\frac{1}{2}]$ ($[t]$ is the integral part of t).

Putting $U(\cdot,\varepsilon)$ and $\tau_0 + \varepsilon\tau_1$ in problem (1) instead of $u(\cdot,\varepsilon)$ and $\lambda(\varepsilon)$ respectively and estimating the residuals, we deduce the inequality

$$\|U(\cdot,\varepsilon) - (1+\tau_0+\varepsilon\tau_1)\,A_\varepsilon U(\cdot,\varepsilon)\|_{H^1(\Omega_\varepsilon)} \leqslant c\varepsilon^{3/2}, \qquad (51)$$

where the constant c is independent of ε. The first part of Lemma 12 in [12] yields the estimate

$$\min_{m\in\mathbb{N}}\Big|\frac{1}{1+\tau_0+\varepsilon\tau_1} - \frac{1}{1+\lambda_m(\varepsilon)}\Big| \leqslant \frac{\|A_\varepsilon U - (1+\tau_0+\varepsilon\tau_1)^{-1}U\|_\varepsilon}{\|U\|_\varepsilon} \leqslant c\varepsilon^{3/2}, \qquad (52)$$

which partly justifies the asymptotics constructed above for the solutions of the spectral problem (1).

3.2. The case $\tau_0 \in \sigma_{\mathbf{D}} \setminus \sigma_{\Omega_0}$. In the previous subsections everything has been prepared in order to determine terms of the asymptotic expansions for the eigenvalue $\lambda(\varepsilon)$ and eigenfunction $u(\cdot,\varepsilon)$ of the original perturbed problem (1) for this case. But, as distinct from before considered case, the outer and inner asymptotic expansions for the eigenfunction are expansions with respect to the scale $\{\varepsilon^{k-\frac{1}{2}} : k \in \mathbb{N}_0\}$, and the eigenvalue $\lambda(\varepsilon)$ is expanded in an integer power series about ε.

Let τ_0 be an eigenvalue of problem (27). Then $v^+_{-\frac{1}{2}}(x^0) = \gamma_{-\frac{1}{2}}(x_1)\varphi(x_2)$, $x^0 \in D$, the corresponding eigenfunction, where φ is defined by formula (29). The $\gamma_{-\frac{1}{2}}$ is involved in the determination of the first term $v^-_{\frac{1}{2}}$ of the outer expansion (10), namely,

$$v^-_{\frac{1}{2}}(x) = |\omega|\,\sqrt{\tau_0}\,\mathbb{U}(x;\gamma_{-\frac{1}{2}}), \quad x \in \Omega_0,$$

where $\mathbb{U}$ is defined by (38).

For the function $V_{\frac{1}{2}}$ we obtain the following problem

$$-\partial^2_{x_2} V_{\frac{1}{2}}(x^0) - \tau_0 V_{\frac{1}{2}}(x^0) \;=\; \tau_1 v^+_{-\frac{1}{2}}(x^0), \quad x_2 \in (0,l);$$

$$V_{\frac{1}{2}}(x_1,0) = |\omega|\sqrt{\tau_0}\Big(\big(J\gamma_{-\frac{1}{2}}\big)(x_1) - \gamma_{-\frac{1}{2}}(x_1)\big(\pi^{-1}\ln\varepsilon + c_\omega\big)\Big), \quad \partial_{x_2}V_{\frac{1}{2}}(x_1,l) = 0.$$

The solvability condition for this problem reads as follows

$$\tau_1\,\gamma_{-\frac{1}{2}}(x_1) \;=\; -\,\frac{2|\omega|}{l}\,\tau_0\,\Big(\big(J\gamma_{-\frac{1}{2}}\big)(x_1) \;-\; \gamma_{-\frac{1}{2}}(x_1)\big(\pi^{-1}\ln\varepsilon \,+\, c_\omega\big)\Big), \quad x_1 \in I_h. \tag{53}$$

Relation (53) is the spectral problem for the operator $J : H^{\frac{1}{2}}_{\ln}(I_h) \mapsto H^{-\frac{1}{2}}_{\ln}(I_h)$, which is continuous, symmetric, and discrete (see (39)). Thus $\gamma_{-\frac{1}{2}} = b_k$ and

$$\tau_1 \;=\; \tau_1^{(k)} \;=\; 2\,|\omega|\,l^{-1}\tau_0\,\big(\pi^{-1}\ln\varepsilon + c_\omega \,-\, \Lambda_k\big), \quad k \in \mathbb{N}, \tag{54}$$

where Λ_k is an eigenvalue and b_k is the corresponding eigenfunction of the operator J.

Now, similarly as in § 3.1, we construct a global asymptotic approximation $U(\cdot,\varepsilon)$ belonging to $H^1(\Omega_\varepsilon)$ for the eigenfunction $u(\cdot,\varepsilon)$:

$$\begin{aligned} U(x,\varepsilon) \;=\; \varepsilon^{\frac{1}{2}}\Big(&(1-\chi(r/\varepsilon))\,v^-_{\frac{1}{2}}(x) + \chi(r)\,\big(V_{\frac{1}{2}}(x_1,0) + \sqrt{\tau_0}\,|\omega|\,b_k(x_1)\,W_2(\xi)\big) - \\ &- \chi(r)\,(1-\chi(r/\varepsilon))\,|\omega|\,\sqrt{\tau_0}\,b_k(x_1)\,\big(\Lambda_k \,-\, \pi^{-1}\ln r\,\big)\Big), \quad x \in \Omega_0; \end{aligned} \tag{55}$$

$$\begin{aligned} U(x,\varepsilon) = \varepsilon^{-\frac{1}{2}} v^+_{-\frac{1}{2}}(x_1,x_2) + \varepsilon^{\frac{1}{2}}\Big(&V_{\frac{1}{2}}(x_1,x_2) + Y_1(\tfrac{x_1}{\varepsilon})\,\partial_1 v^+_{-\frac{1}{2}}(x_1,x_2) + \\ &+ \chi(x_2)\,\sqrt{\tau_0}\,b_k(x_1)\,\big(|\omega| W_2(x/\varepsilon) \,-\, \varepsilon^{-1}x_2\big)\Big), \quad x \in G_\varepsilon, \end{aligned} \tag{56}$$

and prove estimates similarly to the estimates (51) and (52).

3.3. Justification. To justify the constructed asymptotic approximations we use the scheme proposed in [9]. The basic spaces and the corresponding operators are specified in subsections 1.2 and 2.4. Special extension operator is constructed similarly as in [8]. Conditions C5 and C6, in fact, has been verified in the previous subsections: the result of the action of the operator R_ε is the construction of the approximation function U on the basis of an eigenfunction of the limit spectral problem (30). Applying this scheme to problems (1) and (30), we get the following theorems.

THEOREM 1. *Let $\tau_0 \in \sigma_{\Omega_0} \setminus \sigma_D$ and its multiplicity is equal to q; $\tau_1^{(i)}$, $i = 1, \dots, q$, are eigenvalues of (48). Then there exist exactly q eigenvalues of problem (1) with the following asymptotics*

$$\lambda^{(i)}(\varepsilon) \sim \tau_0 + \varepsilon\tau_1^{(i)} + \mathcal{O}(\varepsilon^{3/2}), \quad i = 1, \dots, q.$$

For the corresponding eigenfunctions we have the following estimates

$$\|u^{(i)}(\cdot,\varepsilon) - U^{(i)}(\cdot,\varepsilon)\|_{H^1(\Omega_\varepsilon)} \le C_i\varepsilon^{3/2}, \quad i = 1, \dots, q,$$

where $U^{(i)}$ is defined by (49) and (50).

THEOREM 2. *Let $\tau_0 \in \sigma_D \setminus \sigma_{\Omega_0}$; $\tau_1^{(k)}(\ln\varepsilon)$, $k \in \mathbb{N}$, are defined by (54). Then for any there exists an eigenvalues $\lambda^{(k)}(\varepsilon)$ of problem (1) with the following asymptotics*

$$\lambda^{(k)}(\varepsilon) \sim \tau_0 + \varepsilon\tau_1^{(k)}(\ln\varepsilon) + \mathcal{O}(\varepsilon^{3/2}), \quad k \in \mathbb{N}.$$

Let Λ_k be a simple eigenvalue of the integral operator J. Then

$$\|u^{(k)}(\cdot,\varepsilon) - U^{(k)}(\cdot,\varepsilon)\|_{H^1(\Omega_\varepsilon)} \le C_k\varepsilon^{3/2},$$

where $U^{(k)}$ is defined by (55) and (56).

THEOREM 3. (THE HAUSDORFF CONVERGENCE) *Only the points of the spectrum of problem (30) are accumulation points for the spectrum of problem (1) as $\varepsilon \to 0$.*

The eigenvalues $\{\lambda_n(\varepsilon)\}$ at fixed indices n, are usually called *low eigenvalues*; the corresponding proper-oscillations are called *low frequency oscillations.*

DEFINITION 1. The value $\mathcal{T} := \sup_{n\in\mathbf{N}} \overline{\lim}_{\varepsilon\to 0} \lambda_n(\varepsilon)$ is called the threshold of low eigenvalues of problem (1).

THEOREM 4. (LOW-FREQUENCY CONVERGENCE) *The threshold $\mathcal{T} = \pi^2/4l^2$.*

Let $\mu_0 < \mu_1 \le \dots \le \mu_{m_0}$ be eigenvalues of problem (13), which are less than $\mathcal{T}$, and $\mu_{m_0+1} \ge \mathcal{T}$. Then $\forall\, n = 1, 2, \dots, m_0$

$$\lambda_n(\varepsilon) \to \mu_n \quad as \quad \varepsilon \to 0.$$

For any $n \ge m_0$

$$\lambda_n(\varepsilon) \to \mathcal{T} \quad as \quad \varepsilon \to 0.$$

ACKNOWLEDGMENT. The author is grateful to the Alexander von Humboldt Foundation and Prof. W. L. Wendland for the possibility to carry out this research at the University of Stuttgart. Also I would like to thank Prof. S.A. Nazarov for the discussion and advises.

1. *Argatov I.I. and Nazarov S.A.* Asymptotic analysis of problems on junctions of domains of different limit dimensions. A body pierced by a thin rod. // Izv. Ross. Akad. Nauk, Ser. Mat.- 60.- 1996.- No.1.- 3-36 (in Russian).
2. *Fedoryuk M.V.* The Dirichlet problem for the Laplace operator in the exterior of a thin solid of revolution // Theory of Cubature Formulas and the Application of Functional Analysis to Problems of Mathematical Physics.- Trudy Sem. S.L. Soboleva. Akad. Nauk SSSR Sibirsk. Otdel.- Inst. Mat.- Novosibirsk.- 1.- 1980.- 113-131 (in Russian).
3. *Maz'ya V. G., Nazarov S.A., Plamenevskii B.A.* The asymptotic behavior of solutions of the Dirichlet problem in a domain with a cut out thin tube // Mat.Sb.- 116(158).- 1981.- No. 2.- 187-217 (in Russian).
4. *Mel'nyk T.A. and Nazarov S.A.* Asymptotic structure of the spectrum of the Neumann problem in a thin comb-like domain // C.R. Acad Sci. Paris.- 319.- 1994.- Serie 1.- 1343-1348.
5. *Mel'nyk T.A. and Nazarov S.A.* Asymptotics of the Neumann spectral problem solution in a domain of "thick comb"type // Trudy Seminara imeni I.G. Petrovskogo, Moscow University.- 19.- 1996.- 138-173 (in Russian); English transl. in: Journal of Mathematical Sciences.- 85.- 1997.- No. 6.- 2326-2346.
6. *Mel'nyk T.A.* Homogenization of the Poisson equation in a thick periodic junction // Zeitschrift für Analysis und ihre Anwendungen.- 18.- 1999.- No. 4.- 953-975.
7. *Mel'nyk T.A.* Asymptotic analysis of a spectral problem in a periodic thick junction of type 3:2:1 // Math. Methods in the Applied sciences.- 23.- 2000.- No. 4.- 321-346.
8. *Mel'nyk T.A. and Nazarov S.A.* Asymptotic analysis of the Neumann problem of the junction of a body and thin heavy rods // St.Petersburg Math.J.- 12.- 2001.- No. 2.- 317-351.
9. *Mel'nyk T.A.* Hausdorff convergence and asymptotic estimates of the spectrum of a perturbed operator // Zeitschrift für Analysis und ihre Anwendungen.- 20.- 2001.- No. 4.- 941-957.
10. *Nazarov S.A.* Averaging of boundary value problems in a domain that contains a thin cavity with periodically changing cross-section // Trudy Moskov.Mat.Obshch.- 53.- 1990.- 98-129. (in Russian).
11. *Nazarov S.A.* Junctions of singularly degenerating domains with different limit dimensions, Part II // Trudy Seminara imeni I.G. Petrovskogo, Moscow University. -20.- 1995.- 154-195. (in Russian).
12. *Vishik M.I., Lyusternik L.A.* Regular degeneration and boundary layer for linear differential equations with a parameter // Uspekhi Mat. Nauk.- 12.- 1957.- No. 5.- 3-192 (in Russian).

National Taras Shevchenko University of Kyiv,
Faculty of Mathematics and Mechanics,
Volodymyrska str. 64,
01033 Kyiv, Ukraine
melnyk@imath.kiev.ua

Received 31.08.2004